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A NOTE ON $r-partitions$ OF n IN WHICH THE LEAST PART IS k

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ABSTRACT

Partitions play an important role in Number Theory. It has wide applications in various fields. An attempt is made to develop a theorem on the number of $r-partiions$ of positive integer n in which the least part is k, a reduction theorem on $r-partiions$ and some more results on $r-partiions$ are derived.

Key words: $p(n)$, $r-partiions$, $p_r(n)$, $p_r(e;n)$, $p_r(o;n)$ and $p_r(S;n)$
Subject classification: 11P81 Elementary theory of partitions.

1. Introduction:

1.1 Partition: A partition of a positive integer n is a finite sequence of non-increasing positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$

The number λ_i is called the i^{th} part of the partition. The partition is also denoted as $n = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.

1.2 Partition function: The partition function $p(n)$ is the number of partitions of n.

1.3 r-partition*:* A partition containing r parts is called $r-partiions$.

1.4 $p_r(n)$: $p_r(n)$ is the number of $r-partiions$ of a positive integer n.

Note: $p(n) = p_1(n) + p_2(n) + \ldots + p_n(n)$

1.5 $p_r(o;n)$: $p_r(o;n)$ is the number of partitions of a positive integer n having r parts in which each part is odd number.

1.6 $p_r(e;n)$: $p_r(e;n)$ is the number of partitions of a positive integer n having r parts in which each part is even number.

1.7 $p_r(S;n)$: $p_r(S;n)$ is the number of partitions of a positive integer n having r parts in which each part is the element of the set S.

2. Theorem: Let $r,n \in N(r \le n)$ and $S = \{am+b \mid a \in N, b \in Z \ and \ m = 1,2,\ldots,n\}$ be the set of positive integers. If $a \mid n-br$, then $p_r(S;n) = p_r\left(\frac{n-br}{a}\right)$ other wise $p_r(S;n) = 0$.

$[2.1]$

Proof: All parts in $r-partiions$ of n multiplied by a and added by b to get the partitions of n whose parts are elements of S.

3. Theorem: Let $r,n \in N$ and $S = \{am+b \mid a \in N, b \in Z \ and \ m = 1,2,\ldots,n\}$ be the set of positive integers. Then, the highest least part of $r-partiions$ of n in which the parts are the elements of the set S is $a\left[\frac{n}{ar+b}\right]+b$

[3.1]

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the first, second,…,
r^{th} parts of the r-partition of 'n' respectively. So $n = (\lambda_1, \lambda_2, \ldots, \lambda_r)$

All the distinct $r-partiions$ of n are arranged in such a way that all the parts and corresponding parts in each $r-partiions$ are monotonically increasing.

If possible, let $\lambda_1 = \left\{ a\left[\dfrac{n}{ar+b}\right]+b \right\}+1$

Since all the parts in each $r-partiions$ are monotonically increasing order, the least possible value of each λ_i for i = 2 to r is

$$\left\{ a\left[\frac{n}{ar+b}\right]+b \right\}+1 .$$

Then the sum of all parts in partition is

$$r\left\{\left\{ a\left[\frac{n}{ar+b}\right]+b \right\}+1\right\}.$$

But $r\left\{\left\{ a\left[\dfrac{n}{ar+b}\right]+b \right\}+1\right\} > n$

This is contradiction.

Hence $\lambda_1 = a\left[\dfrac{n}{ar+b}\right]+b$ is the highest integer.

4. Theorem: Let $r, n \in N$ and $S = \{am+b \mid a \in N, b \in Z \ and \ m = 1,2,\ldots,n\}$ be the set of positive integers. Then prove that

$$p_r(S;n) - p_{r-1}\left(S;n-\{a(1)+b\}\right) = p_r(S;n-ar) \qquad [4.1]$$

Proof:
The number of $r-partiions$ of n whose parts are elements of S with least part $a(1)+b$ is equal to the number of $(r-1)-partitions$ of $n-\{a(1)+b\}$ whose parts are elements of S and the number of $r-partiions$ of n whose parts are elements of S with least part is not $a(1)+b$ is equal to the number of $r-partiions$ of $n-ar$ whose parts are elements of S .

$$\therefore p_r(S;n) - p_{r-1}\left(S;n-\{a(1)+b\}\right) = p_r(S;n-ar)$$

5. Theorem: Let $r, n, k \in N$ and $S = \{am+b \mid a \in N, b \in Z \ and \ m = 1,2,\ldots,n\}$ be the set of positive integers. then, the number of $r-partiions$ of n having the parts are elements of S with least part k is

$$p_{r-1}\left(S;n-(k-1)ar-\{a+b\}\right)$$

where $1 \le k \le \left[\dfrac{n}{ar+b}\right]$

[5.1]

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the first, second,…,
r^{th} parts of the $r-partiions$ of n respectively. So

$$n = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

All the distinct $r-partiions$ of n are arranged in such a way that all the parts and corresponding parts in each $r-partiions$ are monotonically increasing.

Fixing $\lambda_1 = a(1)+b$, the remaining value $n-\{a(1)+b\}$ of n can be expressed as the sum of the remaining $r-1$ parts $\lambda_2, \lambda_3, \ldots, \lambda_r$ in $p_{r-1}\left(S;n-\{a(1)+b\}\right)$ ways.

i,e The number of $r-partiions$ in which the least part of the partition is $\lambda_1 = a(1)+b$ is

$$p_{r-1}\left(S;n-\{a+b\}\right).$$

Fixing $\lambda_1 = a(2)+b$, the remaining value $n-\{a(2)+b\}$ of n can be expressed as the sum of the remaining $r-1$ parts $\lambda_2, \lambda_3, \ldots, \lambda_r$ in $p_{r-1}\left(S;n-\{a(2)+b\}\right)$ ways.

Since all the parts in each r-partition are non decreasing, $p_{r-2}\left(S;n-\{a(3)+b(2)\}\right)$ $r-partiions$ with $\lambda_1 = a(2)+b$, $\lambda_2 = a(1)+b$ are to be eliminated from $p_{r-1}\left(S;n-\{a(2)+b\}\right)$ $r-partiions$ .Then, the number of the $r-partiions$ in which the least part of the partition $\lambda_1 = a(2)+b$ is

$p_{r-1}\left(S;n-\{a(2)+b\}\right)$

-

$p_{r-2}\left(S;n-\{a(3)+b(2)\}\right)$

$= p_{r-1}\left(S;n-a(r-1)-\{a(2)+b\}\right)$

$= p_{r-1}\left(S;n-ar-\{a+b\}\right)$

Fixing $\lambda_1 = a(3)+b$, the remaining value $n-\{a(3)+b\}$ of n can be expressed as the sum of the remaining $r-1$ parts $\lambda_2, \lambda_3, \ldots, \lambda_r$ in $p_{r-1}\left(S;n-\{a(3)+b\}\right)$ ways.

$p_{r-2}\left(S;n-\{a(4)+b(2)\}\right)$ $r-partiions$ with $\lambda_1 = a(3)+b$, $\lambda_2 = a(1)+b$ and

$p_{r-2}\left(S;n-\{a(5)+b(2)\}\right)$

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$p_{r-3}\left(S;n-\{a(6)+b(3)\}\right)$

$r-partiions$ with $\lambda_1 = a(3)+b$, $\lambda_2 = a(2)+b$ are to be eliminated from $p_{r-1}\left(S;n-\{a(3)+b\}\right)$ $r-partiions$.

Then, the number of the $r-partiions$ in which the least part of the partition $\lambda_1 = a(3)+b$ is

$p_{r-1}\left(S;n-\{a(3)+b\}\right)$

- $p_{r-2}\left(S;n-\{a(4)+b(2)\}\right)$

-

$$\left\{\begin{aligned} &p_{r-2}\left(S;n-\{a(5)+b(2)\}\right) \\ &\quad - p_{r-3}\left(S;n-\{a(6)+b(3)\}\right)\end{aligned}\right\}$$

$$\begin{aligned}
&= p_{r-1}\left(S;n-a(r-1)-\{a(3)+b\}\right) \\
&\qquad - p_{r-2}\left(S;n-a(r-2)-\{a(5)+b\}\right) \\
&= p_{r-1}\left(S;n-ar-\{a(2)+b\}\right) \\
&\qquad - p_{r-2}\left(S;n-ar-\{a(3)+b\}\right) \\
&= p_{r-1}\left(S;n-a(r-1)-ar-\{a(2)+b\}\right) \\
&= p_{r-1}\left(S;n-2ar-\{a+b\}\right)
\end{aligned}$$

By induction we observe that the number of $r-partiions$ of n having the parts are elements of S with least part k is

$$p_{r-1}\left(S;n-(k-1)ar-\{a+b\}\right)$$

where $1 \le k \le \left[\dfrac{n}{ar+b}\right]$

Corollary 5.1: *Let* $n,r,k \in N$. Then the number of $r-partiions$ of n with least part k is

$p_{r-1}\left[n-(k-1)r-1\right]$ where $0 \le k \le \left[\dfrac{n}{r}\right]$

Proof: Put $a=1, b=0$ in $[5.1]$

Corollary 5.2: *Let* $n,r,k \in N$. Then, the number of $r-partiions$ of n having the parts are even numbers with least part k is

$p_{r-1}\left[e;n-2(k-1)r-2\right]$ where $0 \le k \le \left[\dfrac{n}{2r}\right]$

Proof: Put $a=2, b=0$ in $[5.1]$

Corollary 5.3: *Let* $n,r,k \in N$. Then the number of $r-partiions$ of n having the parts are odd numbers with least part k is

$$p_{r-1}\left[o;n-2(k-1)r-1\right]$$

where $0 \le k \le \left[\dfrac{n}{2r-1}\right]$

Proof: Put $a=2, b=-1$ in $[5.1]$

6. Reduction theorem for $p_r(S;n)$:

Let $r,n,k \in N$ and

$S = \{am+b \mid a \in N, b \in Z \text{ and } m = 1,2,\ldots,n\}$

be the set of positive integers. then,

$$p_r(\mathrm{S};n) = \sum_{k=1}^{\left[\frac{n}{ar+b}\right]} p_{r-1}\left(S;n-(k-1)ar-\{a+b\}\right)$$

[6.1]

and

$$p(\mathrm{S};n) = \sum_{r=1}^{n}\sum_{k=1}^{\left[\frac{n}{ar+b}\right]} p_{r-1}\left(S;n-(k-1)ar-\{a+b\}\right)$$

[6.2]

Proof: From $[5.1]$ we can observe it

Corollary 6.1: Prove that

$$p(n) = \sum_{r=1}^{n}\sum_{k=1}^{\left[\frac{n}{r}\right]} p_{r-1}\left(n-(k-1)r-1\right)$$

Proof: Put $a=1, b=0$ in $[6.2]$

Corollary 6.2: Prove that

$$p(\mathrm{e};n)=\sum_{r=1}^{n}\sum_{k=1}^{\left[\frac{n}{2r}\right]} p_{r-1}\left(e;n-2(k-1)r-2\right)$$

Proof: Put $a=2, b=0$ in $[6.2]$

Corollary 6.3: Prove that

$$p(\mathrm{o};n)=\sum_{r=1}^{n}\sum_{k=1}^{\left[\frac{n}{2r-1}\right]} p_{r-1}\left(o;n-2(k-1)r-1\right)$$

Proof: Put $a=2, b=-1$ in $[6.2]$

Theorem 7: *If* $r,n\in N$ *and* $r<n$, then

$$p_r(n)=p(n-r) \quad \text{for} \quad \frac{n}{r}\le 2$$

Proof:

Case:1 Let $\frac{n}{r}=2$

$$\Rightarrow n=2r$$

$$\begin{aligned}
p_r(n) &= p_1(n-r)+p_2(n-r)+\dots+p_r(n-r)\\
&= p_1(r)+p_2(r)+\dots+p_r(r)\\
&= p(r)\\
&= p(n-r)
\end{aligned}$$

Case:2 Let $\frac{n}{r}<2$

Since $r<n$ and $\frac{n}{r}<2$

$$\Rightarrow r<n<2r$$
$$\Rightarrow 0<n-r<n$$

$$\begin{aligned}
p_r(n) &= p_1(n-r)+p_2(n-r)+\dots\\
&\quad +p_{n-r}(n-r)+p_{n-r+1}(n-r)+\dots+p_r(n-r)\\
&= p_1(n-r)+p_2(n-r)+\dots\\
&\quad +p_{n-r}(n-r)+0+\dots+0\\
&= p(n-r)
\end{aligned}$$

Hence $p_r(\mathrm{n})=p(n-r)$ for $\frac{n}{r}\le 2$

Theorem 8: Let $n,i,j\in N$, then

$$p(n)=1+\sum_{i+j=2}^{n} p_i(j)$$

Proof: Case 1: Let $n=2m$ *for* $m\in N$

$$\begin{aligned}
p(n) &= p(2m)\\
&= p_1(2m)+p_2(2m)+p_3(2m)+\dots\\
&\quad +p_{m-1}(2m)+p_m(2m)+p_{m+1}(2m)+\dots\\
&\quad +p_{2m-2}(2m)+p_{2m-1}(2m)+p_{2m}(2m)
\end{aligned}$$

$$\begin{aligned}
&=\{p_1(2m-1)\}\\
&+\{p_1(2m-2)+p_2(2m-2)\}\\
&+\{p_1(2m-3)+p_2(2m-3)+p_3(2m-3)\}+\dots\\
&+\{p_1(m+1)+p_2(m+1)+\dots+p_{m-1}(m+1)\}\\
&+p(m)+p(m-1)+\dots+p(2)+p(1)+1
\end{aligned}$$

$$\begin{aligned}
&=\{p_1(2m-1)\}\\
&+\{p_1(2m-2)+p_2(2m-2)\}\\
&+\{p_1(2m-3)+p_2(2m-3)+p_3(2m-3)\}+\dots\\
&+\{p_1(m+1)+p_2(m+1)+\dots+p_{m-1}(m+1)\}\\
&+\{p_1(m)+p_2(m)+\dots+p_{m-1}(m)+p_m(m)\}\\
&+\{p_1(m-1)+p_2(m-1)+\dots+p_{m-1}(m-1)\}+\dots\\
&+\{p_1(3)+p_2(3)+p_3(3)\}\\
&+\{p_1(2)+p_2(2)\}+p_1(1)+1
\end{aligned}$$

$$\begin{aligned}
&=1+\sum_{i+j=2m} p_i(j)+\sum_{i+j=2m-1} p_i(j)+\dots\\
&\quad +\sum_{i+j=3} p_i(j)+\sum_{i+j=2} p_i(j)\\
&=1+\sum_{i+j=2}^{2m} p_i(j)\\
&=1+\sum_{i+j=2}^{n} p_i(j)
\end{aligned}$$

Case 2: Let $n=2m+1$ *for* $m\in N$

$$\begin{aligned}
p(n) &= p(2m+1)\\
&= p_1(2m+1)+p_2(2m+1)\\
&\quad +p_3(2m+1)+\dots+p_m(2m+1)\\
&\quad +p_{m+1}(2m+1)+p_{m+2}(2m+1)+\dots\\
&\quad +p_{2m-1}(2m+1)+p_{2m}(2m+1)\\
&\quad +p_{2m+1}(2m+1)
\end{aligned}$$

$=\{p_1(2m)\}$
$+\{p_1(2m-1)+p_2(2m-1)\}$
$+\{p_1(2m-2)+p_2(2m-2)+p_3(2m-2)\}+\dots$
$+\{p_1(m+1)+p_2(m+1)+\dots+p_m(m+1)\}+$
$p(m)+p(m-1)+\dots+p(2)+p(1)+1$
$=\{p_1(2m)\}$
$+\{\ p_1(2m-1)+p_2(2m-1)\}$
$+\{\ p_1(2m-2)+p_2(2m-2)+p_3(2m-2)\}+\dots$
$+\{\ p_1(m+1)+p_2(m+1)+\dots+p_m(m+1)\}$
$+\{\ p_1(m)+p_2(m)+\dots+p_m(m)\}$
$+\{\ p_1(m-1)+p_2(m-1)+\dots+p_{m-1}(m-1)\}+\dots$
$+\{\ p_1(3)+p_2(3)+p_3(3)\}$
$+\{\ p_1(2)+p_2(2)\}+p_1(1)+1$

$$=1+\sum_{i+j=2m+1} p_i(j)+\sum_{i+j=2m} p_i(j)+\dots+$$

$$=1+\sum_{i+j=2}^{2m+1} p_i(j)$$

$$=1+\sum_{i+j=2}^{n} p_i(j)$$

Hence $\quad p(n)=1+\sum_{i+j=2}^{n} p_i(j)$

Corollary 8.1: Let n be a natural number, then

$$p(n)-p(n-1)=\sum_{i+j=n} p_i(j)$$

Proof: Since $p(n)=1+\sum_{i+j=2}^{n} p_i(j)$

$$\therefore\ p(n-1)=1+\sum_{i+j=2}^{n-1} p_i(j)$$

Hence $\quad p(n)-p(n-1)=\sum_{i+j=n} p_i(j)$

Theorem 9: If $n\in N$ then

$$p(n)=\begin{cases} 1+p_{\frac{n}{2}-1}\left(\dfrac{3n}{2}-1\right)+\displaystyle\sum_{i=1}^{\frac{n}{2}} p(i) & \text{when } n \text{ is even} \\ 1+p_{\frac{n-1}{2}}\left(\dfrac{3n-1}{2}\right)+\displaystyle\sum_{i=1}^{\frac{n-1}{2}} p(i) & \text{when } n \text{ is odd}\end{cases}$$

Proof:
If n is even
Let $n=2m\quad for\quad m\in N$

$$p(n)=\ p_1(n)+p_2(n)+\dots+p_{m-1}(n)+p_m(n)$$
$$+\ p_{m+1}(n)+\dots+p_{2m\text{-}2}(n)+p_{2m-1}(n)+p_{2m}(n)$$

$$=p_1(2m)+p_2(2m)+\dots+p_{m-1}(2m)+p_m(2m)$$
$$+\ p_{m+1}(2m)+\dots+p_{2m\text{-}2}(2m)+p_{2m-1}(2m)$$
$$+\ p_{2m}(2m)$$

$$=p_{m-1}(2m+m-1)+p(m)+p(m\text{-}1)+\dots+p(2)$$
$$+\ p(1)+1$$

$$=1+p_{m-1}(2m+m\text{-}1)+\{p(1)+p(2)+\dots+p(m)\}$$

$$=1+p_{m-1}(3m\text{-}1)+\sum_{i=1}^{m} p(i)$$

$$=1+p_{\frac{n}{2}-1}\left(\frac{3n}{2}\text{-}1\right)+\sum_{i=1}^{\frac{n}{2}} p(i)$$

If n is odd
Let $n=2m+1\quad for\quad m\in N$

$$p(n)=\ p_1(n)+p_2(n)+\dots+p_{m-1}(n)$$
$$+\ p_m(n)+p_{m+1}(n)+p_{m+2}(n)+\dots$$
$$+\ p_{2m-1}(n)+p_{2m}(n)+p_{2m+1}(n)$$

$$=p_1(2m+1)+p_2(2m+1)+\dots+p_m(2m+1)$$
$$+\ p_{m+1}(2m+1)+p_{m+2}(2m+1)+\dots$$

$$+\ p_{2m\text{-}1}(2m+1)+p_{2m}(2m+1)+p_{2m+1}(2m+1)$$
$$=p_m(2m+m+1)+p(m)+p(m\text{-}1)+\dots$$
$$+\ p(2)+p(1)+1$$

$$=1+p_m(3m+1)+\sum_{i=1}^{m} p(i)$$

$$=1+p_{\frac{n-1}{2}}\left(\frac{3n\text{-}1}{2}\right)+\sum_{i=1}^{\frac{n-1}{2}} p(i)$$

Hence

$$p(n)=\begin{cases} 1+p_{\frac{n}{2}-1}\left(\dfrac{3n}{2}-1\right)+\displaystyle\sum_{i=1}^{\frac{n}{2}} p(i) & \text{when } n \text{ is even} \\ 1+p_{\frac{n-1}{2}}\left(\dfrac{3n-1}{2}\right)+\displaystyle\sum_{i=1}^{\frac{n-1}{2}} p(i) & \text{when } n \text{ is odd}\end{cases}$$

Theorem 10:
If $\left[\dfrac{m}{r}\right]=1$,then $\left[\dfrac{m+1}{r+1}\right]=1$ for $m,r\in Z$

Proof: Since $\left[\dfrac{m}{r}\right]=1$

$$\Rightarrow r\le m<2r$$
$$\Rightarrow r+1\le m+1<2r+1$$

Since $r+1 \le m+1 < 2r+1$ *and* $2r+1 \le 2r+2$

$$\therefore r+1 \le m+1 < 2r+2$$

$$\Rightarrow 1 \le \frac{m+1}{r+1} < 2$$

$$\Rightarrow \left[\frac{m+1}{r+1}\right] = 1$$

Hence $\left[\frac{m}{r}\right] = 1 \Rightarrow \left[\frac{m+1}{r+1}\right] = 1$ for $m, r \in Z$

Theorem 11: If $\left[\frac{m}{r}\right] = 1$, then

$$p_r(m) + p_r(m+r+1) + ... + p_r(m+tr+t)$$

$$= p_{r+1}(m+tr+t)$$

Proof: Let $t = 1$

Since $\left[\frac{m}{r}\right] = 1$

There fore

$$\left[\frac{m+r+1+1}{r+1}\right] = \left[\frac{m+1}{r+1} + \frac{r+1}{r+1}\right] = [1+1] = 2$$

Since

$$p_{r+1}(m+r+1+1) = p_r(m+r+1)$$

$$+\left[p_r(m+r) - p_r(m+r-1)\right]$$

$$= p_r(m+r+1) + p_r(m+r-r)$$

$$= p_r(m+r+1) + p_r(m)$$

We assume that it is true for $t = s$

There fore

$$p_r(m) + p_r(m+r+1) + ... + p_r(m+sr+s)$$

$$= p_{r+1}(m+sr+s+1)$$

Adding both sides by $p_r(m+sr+s+r+1)$

$$p_r(m) + p_r(m+r+1) + ...$$

$$+ p_r(m+sr+s) + p_r(m+sr+s+r+1)$$

$$= p_{r+1}(m+sr+s+1) + p_r(m+sr+s+r+1)$$

$$= p_{r+1}\left[m+(r+1)s+1\right] + p_r\left[m+(r+1)(s+1)\right]$$

$$= p_{r+1}\left[m+(r+1)s+1\right] + p_r\left[m+(r+1)s+1+r\right]$$

$$= p_{r+1}\left[m+(r+1)(s+1)+1\right]$$

It is true for $t = s+1$

There fore our statement is true for all $t \in N$

Hence

$$p_r(m) + p_r(m+r+1) + ... + p_r(m+tr+t)$$

$$= p_{r+1}(m+tr+t) \;\; when \left[\frac{m}{r}\right] = 1$$

Corollary 11.1: Prove that

$$p_r(r) + p_r(2r+1) + ... + p_r\left[nr+(n-1)\right]$$

$$= p_{r+1}\left[n(r+1)\right]$$

Proof:

From theorem 11

$$p_r(m) + p_r(m+r+1) + ... + p_r(m+tr+t)$$

$$= p_{r+1}(m+tr+t) \;\; when \left[\frac{m}{r}\right] = 1$$

Put $m = r$ and

$$p_r(r) + p_r(2r+1) + ... + p_r\left[nr+(n-1)\right]$$

$$= p_{r+1}\left[n(r+1)\right]$$

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REFERENCES

[1]. George E. Andrews, The theory of partitions, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[2]. G. E. Andrews and F. G. Garvan, Dyson’s crank of a partition, Bull. Amer. Math. Soc. (N.S.) 18 (1988), 167–171.

[3]. G. E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math., to appear.

[4]. G. E. Andrews and F. G. Garvan, Dyson’s crank of a partition, Bull. Amer. Math. Soc. (N.S.) 18 (1988), 167–171.

[5]. T. M. Apostol, Modular Functions and Dirichlet Series in Number Theory, Springer-Verlag, NewYork, 1976.

[6]. A. O. L. Atkin and F. G. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003),343–366.

[7]. K. Bringmann, F. G. Garvan and K. Mahlburg Partition statistics and quasiweak Maass forms, Internat. Math.Res. Notices, to appear.

[8]. K. Bringmann and K. Ono, Dyson’s ranks and Maass forms, Ann. of Math., to appear.

[9]. K. S. Chua, Explicit congruences for the partition function modulo every prime, Arch. Math. (Basel) 81 (2003).

[10]. F. J. Dyson, “Selected papers of Freeman Dyson with commentary,” Amer. Math. Soc., Providence, RI, 1996.

[11]. F. J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944), 10–15.

[12]. F. Garvan, D. Kim, D. Stanton, Cranks and t-cores, Invent. Math. 101 (1990), 1–17.

[13]. G. Gasper, M. Rahman, Basic hypergeometric series, Encyclopedia of Mathematics and its

applications v.35, Cambridge, 1990.
[14]. K.Hanuma Reddy: Note on convex polygons, proc of Joint Conference on Information
Sciences 2007, USA, 1691-1697.
[15]. G. H. Hardy and EM. Wright "An Introduction to the Theory of Numbers," Oxford Univ. Press, London, 1979.
[16]. S. Ramanujan, "The lost notebook and other unpublished papers," Springer-Verlag, Berlin, 1988.